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A fundamental solution for linear second-order elliptic systems with variable coefficients

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Abstract. A fundamental solution and a Green's function are obtained for a system of second-order elliptic partial differential equations with variable coefficients. Both the fundamental solution and the Green's function are suitable for facilitating the numerical solution of boundary-value problems in a number of practical areas. Some particular areas of application are outlined.

Key words: anisotropic, boundary-integral method, Green's function, inhomogeneous elasticity

1. Introduction

This paper is concerned with boundary-value problems governed by a system of linear second-order partial differential equations with variable coefficients. The solution for this class of boundary-value problems may be written in terms of a boundary-integral equation with the kernel of the equation consisting of a fundamental solution or Green's function for the system. For the boundary-integral equation to be of any practical use for the numerical solution of the boundary-value problems in question, it is necessary to obtain a fundamental solution or Green's fuction in a form which readily yields numerical values. To obtain such a fundamental solution or Green's function for a system with quite general variable coefficients is a difficult task. The purpose of the present paper is to obtain a fundamental solution and a particular Green's function for the system in forms which are suitable for numerical calculations for a restricted but important class of variable coefficients. The solutions obtained may be used in the boundary-integral equation to solve a wide class of practical problems. Three such classes of problems are specifically indicated in the paper.

2. The boundary-value problem

Consider the system of partial differential equations

$$\frac{\partial}{\partial x_j}\left[a_{ijkl}(\mathbf{x})\,\frac{\partial\phi_k(\mathbf{x})}{\partial x_l}\right]=0, \quad \text{for } i=1,2,\ldots,N, \tag{1}$$

where in a domain Ω in R^2 with boundary $\partial\Omega$ the $\phi_k(\mathbf{x})$ for $k = 1, 2, \ldots, N$ with $\mathbf{x} = (x_1, x_2)$ are defined and twice differentiable functions of the dependent variables x_1 and x_2 and where the summation convention applies for all lower case italic subscipts. In Ω the coefficients $a_{ijkl}(\mathbf{x})$ are non-negative twice differentiable functions of x_1 and x_2 which satisfy the symmetry conditions

$$a_{ijkl} = a_{klij} \tag{2}$$

and also sufficient conditions for the system of partial differential equations to be elliptic throughout Ω.

A solution to (1) is sought which is valid in the region Ω with boundary $\partial\Omega$. On $\partial\Omega$ either the dependent variables ϕ_k or the P_i are specified where

$$P_i = a_{ijkl}\frac{\partial \phi_k}{\partial x_l}n_j. \tag{3}$$

where $\mathbf{n}$ denotes the outward pointing normal to the boundary $\partial\Omega$. Further, if P_i is specified at all points of the boundary $\partial\Omega$, then for a well-posed problem it must be such that

$$\int_{\partial\Omega} P_i \mathrm{d}s = 0. \tag{4}$$

3. Boundary-integral equation

A boundary-integral equation for the solution of the problem specified in Section 2 may be readily obtained in the form (see for example Clements [1])

$$\lambda\phi_j(\mathbf{x_0}) = \int_{\partial\Omega}\left[P_i(\mathbf{x})\Phi_{ij}(\mathbf{x},\mathbf{x_0}) - \Gamma_{ij}(\mathbf{x},\mathbf{x_0})\phi_i(\mathbf{x})\right]\mathrm{d}S(\mathbf{x}), \tag{5}$$

where $\mathbf{x}_0 = (\xi_1, \xi_2)$ is the source point, $\lambda = 0$ if $\mathbf{x_0} \notin \Omega$, $\lambda = 1$ if $\mathbf{x_0} \in \Omega$ and $\lambda = \frac{1}{2}$ if $\mathbf{x_0} \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $\mathbf{x_0}$. Also P_i is defined by Equation (3) and $\Phi_{ij}(\mathbf{x},\mathbf{x_0})$ is a fundamental solution of (1) so that it satisfies the system of equations

$$\frac{\partial}{\partial x_j}\left[a_{ijkl}(\mathbf{x})\;\frac{\partial \Phi_{km}}{\partial x_l}\right] = \delta_{im}\delta(\mathbf{x}-\mathbf{x_0}), \tag{6}$$

where δ_{im} denotes the Kronecker delta and δ the Dirac delta function. Also in (5) the $\Gamma_{ij}(\mathbf{x},\mathbf{x_0})$ is given by

$$\Gamma_{im} = a_{ijkl}\frac{\partial \Phi_{km}}{\partial x_l}n_j\ . \tag{7}$$

The usefulness of the boundary-integral Equation (5) for generating a numerical solution of the boundary-value problem of section 2 hinges on the availability of a solution $\Phi_{ij}(\mathbf{x},\mathbf{x_0})$ of (6) in a form which readily yields numerical values. To obtain such a solution for quite general coefficients a_{ijkl} is a difficult task. In the following sections a solution suitable for numerical computations is obtained for a restricted class of coefficients.

4. A fundamental solution

The coefficients in (1) are now required to take the form

$$a_{ijkl}(\mathbf{x}) = a^{(0)}_{ijkl} g(\mathbf{x}), \tag{8}$$

where the $a^{(0)}_{ijkl}$ are constants and $g(x_1, x_2)$ is a twice differentiable function of the variables x_1 and x_2. Also $g(\mathbf{x}) > 0$ in Ω and in addition to the symmetry condition (2) the $a^{(0)}_{ijkl}$ are required to satisfy the condition

$$a^{(0)}_{ijkl} = a^{(0)}_{ilkj}. \tag{9}$$

Equation (6) thus may be written

$$a^{(0)}_{ijkl} \frac{\partial}{\partial x_j}\left[g(\mathbf{x}) \frac{\partial \Phi_{km}}{\partial x_l}\right] = \delta_{im}\delta(\mathbf{x} - \mathbf{x_0}). \tag{10}$$

Consider the transformation

$$\Phi_{km}(\mathbf{x}, \mathbf{x_0}) = g^{-1/2}(\mathbf{x})\Psi_{km}(\mathbf{x}, \mathbf{x_0}). \tag{11}$$

Use of (11) in (10) provides the equation

$$\begin{aligned} g^{1/2} a^{(0)}_{ijkl} \frac{\partial^2 \Psi_{km}}{\partial x_j \partial x_l} \;+\; a^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_j} \frac{\partial \Psi_{km}}{\partial x_l} - a^{(0)}_{ijkl} \frac{\partial g^{1/2}}{\partial x_l} \frac{\partial \Psi_{km}}{\partial x_j} \\ -\; \Psi_{km}\, a^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = \delta_{im}\delta(\mathbf{x} - \mathbf{x_0}), \end{aligned} \tag{12}$$

where by virtue of (9) this equation reduces to

$$g^{1/2} a^{(0)}_{ijkl} \frac{\partial^2 \Psi_{km}}{\partial x_j \partial x_l} - \Psi_{km} a^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = \delta_{im}\delta(\mathbf{x} - \mathbf{x_0}) \;. \tag{13}$$

Thus if

$$g^{1/2}(\mathbf{x}) a^{(0)}_{ijkl} \frac{\partial^2 \Psi_{km}}{\partial x_j \partial x_l} = \delta_{im}\delta(\mathbf{x} - \mathbf{x_0}) \tag{14}$$

and

$$a^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0 \;, \tag{15}$$

then (13) will be satisfied. Hence when $g(\mathbf{x})$ satisfies the system (15) the transformation given by (11) transforms the linear system with variable coefficients (10) to the linear system (14).

Equation (15) consists of a system of N constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. A solution to this system may be written as a linear function of the two independent variables x_1 and x_2. Thus for this solution g may be written in the form

$$g(\mathbf{x}) = (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)^2, \tag{16}$$

where the α_i for $i = 0, 1, 2$ are constants which may be used to fit the coefficients $a_{ijkl}(\mathbf{x}) = a^{(0)}_{ijkl} g(\mathbf{x})$ to given numerical data associated with a particular application.

Although the Equation (15) will always be satisfied by taking $g(\mathbf{x})$ in the form given by (16), it is appropriate to note at this stage that for particular classes of equations of the type (1) the tranformation from (10) to (14) may be achieved for a more general class of functions $g(\mathbf{x})$.

From Equation (14) it follows that

$$a^{(0)}_{ijkl}\,\frac{\partial^2\Psi_{km}}{\partial x_j\partial x_l} = g^{-1/2}(\mathbf{x_0})\delta_{im}\delta(\mathbf{x}-\mathbf{x_0}), \tag{17}$$

where the properties of the delta function have been employed to change the variable $\mathbf{x}$ to $\mathbf{x_0}$ in the first term on the right hand side of (17).

Let $\Psi^*_{ij}(\mathbf{x},\mathbf{x_0})$ denote the fundamental solution of the system (1) with $a_{ijkl}(\mathbf{x}) = a^{(0)}_{ijkl}$ so that $\Psi^*_{ij}(\mathbf{x},\mathbf{x_0})$ satisfies the equation

$$a^{(0)}_{ijkl}\,\frac{\partial^2\Psi^*_{km}}{\partial x_j\partial x_l} = \delta_{im}\delta(\mathbf{x}-\mathbf{x_0}). \tag{18}$$

Now from (17) and (18) it follows that

$$\Psi_{km}(\mathbf{x},\mathbf{x_0}) = g^{-1/2}(\mathbf{x_0})\Psi^*_{km}(\mathbf{x},\mathbf{x_0}) \tag{19}$$

and from (11) and (19) the fundamental solution of (1) satisfying (6) may be written in the form

$$\Phi_{km}(\mathbf{x},\mathbf{x_0}) = g^{-1/2}(\mathbf{x})g^{-1/2}(\mathbf{x_0})\Psi^*_{km}(\mathbf{x},\mathbf{x_0}). \tag{20}$$

The fundamental solution $\Psi^*_{km}(\mathbf{x},\mathbf{x_0})$ which satisfies (18) is given by (see Clements and Rizzo [2] and Clements [1])

$$\Psi^*_{im}(\mathbf{x},\mathbf{x}_0) \;=\; \frac{1}{2\pi}\mathfrak{Re}\left[\sum_{\alpha=1}^{N} A_{i\alpha}N_{\alpha k}\log(z_\alpha - c_\alpha)\right]d_{km}, \tag{21}$$

where $\mathfrak{Re}$ denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$ and $c_\alpha = \xi_1 + \tau_\alpha\xi_2$, where τ_α are the N roots with positive imaginary part of the polynomial of degree $2N$ in τ

$$|a^{(0)}_{i1k1} + a^{(0)}_{i2k1}\tau + a^{(0)}_{i1k2}\tau + a^{(0)}_{i2k2}\tau^2| = 0. \tag{22}$$

The $A_{i\alpha}$ occurring in (21) are the solutions of the system of homogeneous linear algebraic equations

$$\left(a^{(0)}_{i1k1} + a^{(0)}_{i2k1}\tau_\alpha + a^{(0)}_{i1k2}\tau_\alpha + a^{(0)}_{i2k2}\tau_\alpha^2\right)A_{k\alpha} = 0. \tag{23}$$

Also the $N_{\alpha k}$, and d_{km} in (21) are defined by

$$\delta_{ik} \;=\; \sum_{\alpha=1}^{N} A_{i\alpha}N_{\alpha k}, \tag{24}$$

$$L_{ij\alpha} \;=\; (a^{(0)}_{ijk1} + \tau_\alpha a^{(0)}_{ijk2})A_{k\alpha}, \tag{25}$$

$$\delta_{im} \;=\; -\frac{1}{2}\mathrm{i}\sum_{\alpha=1}^{N}\left\{L_{i2\alpha}N_{\alpha k} - \overline{L}_{i2\alpha}\overline{N}_{\alpha k}\right\}d_{km}, \tag{26}$$

where the bar denotes the complex conjugate and i denotes the square root of minus one.

Equation (20) together with (21) provides the fundamental solution for a wide class of important equations which are special cases of the system (1). Examples include the equation governing the static two-dimensional temperature field in a class of inhomogeneous isotropic and anisotropic materials and the equations governing steady-state antiplane, plane and generalised plane deformations of a class of inhomogeneous elastic materials. In all of these cases Equation (20) together with (21) provides a fundamental solution which readily yields numerical values. In particular, once the coefficients $a^{(0)}_{ijkl}$ are known then τ_α, $A_{k\alpha}$, $N_{\alpha j}$ and d_{jm} may be calculated in a systematic way through Equations (22) through (26). The Equation (20) together with (21) then provides a simple analytical form for the fundamental solution from which numerical values may be easily obtained. Furthermore the fundamental solution (20) together with (21) may be readily used in the boundary-integral formulation (5) to obtain the numerical solution of boundary-value problems governed by the appropriate special cases of the system (1).

5. A Green's function

Now it is clear that in the case when the coefficients a_{ijkl} are of the form given by (8) and (9) a solution to (10) may consist of the particular solution (20) plus any solution of the associated homogeneous system (1). Here such a solution is considered in order to provide a Green's function for a particular class of boundary-value problems. In particular the Green's function will be written in the form

$$\Phi_{km} = \Phi^{(1)}_{km} + \Phi^{(2)}_{km}, \tag{27}$$

where $\Phi^{(1)}_{km}$ is given by (20) and (21) and $\Phi^{(2)}_{km}$ will be a solution to (1) chosen so that Φ_{km} satisfies given conditions on a specified boundary in R^2.

Consider a region Ω in R^2 with boundary $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$ where $\partial\Omega_2$ lies along $x_2 = 0$. Here Φ_{km} is chosen to be zero on $\partial\Omega_2$. Image considerations indicate that the appropriate choice of $\Phi^{(2)}_{km}$ is

$$\Phi^{(2)}_{km} = -\frac{1}{2\pi} g^{-1/2}(\mathbf{x}) g^{-1/2}(\mathbf{x_0}) \mathfrak{Re} \left\{ \sum_{\alpha} A_{k\alpha} N_{\alpha q} \sum_{\beta} \overline{A}_{q\beta} \overline{N}_{\beta j} \log(z_\alpha - \overline{c}_\beta) \right\} d_{jm}. \tag{28}$$

As with (20) the expression (28) readily provides numerical values of $\Phi^{(2)}_{km}$. The Green's function given by (27), (20), (21) and (28) is suitable for use in the boundary-integral equation formulation for problems where the region under consideration has a large part of its boundary lying along the line $x_2 = 0$. In such cases use of this Green's function can provide a considerable simplification in the evaluation of the integral along $\partial\Omega_2$ in the boundary-integral Equation (5). It is particularly suitable when the boundary lying along $x_2 = 0$ extends to infinity. It is worth noting that although in this case the Green's function has been chosen to be zero on $x_2 = 0$ the use of the method of superposition may be employed to solve a wide class of problems where a significant part of the boundary lies on $x_2 = 0$ and the dependent variable ϕ_k in (1) is not zero for a relatively small interval on $x_2 = 0$ (see Clements and Jones [3]).

6. Applications

6.1. Plane anisotropic thermostatics

In (1) put $N = 1$, $\phi_1 = T$ and $a_{1i1j} = \lambda_{ij}$ for $i, j = 1, 2$ so that (1) becomes

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}(\mathbf{x})\,\frac{\partial T(\mathbf{x})}{\partial x_j}\right] = 0, \tag{29}$$

which, referred to a Cartesian frame $Ox_1x_2x_3$, is the equation governing the two dimensional static temperature field $T(\mathbf{x})$ in an anisotropic material with conductivity coefficients λ_{ij} which satisfy the symmetry property $\lambda_{ij} = \lambda_{ji}$. In this case (8) and (22) through (26) provide

$$\lambda_{ij}(\mathbf{x}) = \lambda_{ij}^{(0)} g(\mathbf{x}), \tag{30}$$

$$\tau_1 = \frac{-\lambda_{12}^{(0)} + \mathrm{i}\left(\lambda_{11}^{(0)}\lambda_{22}^{(0)} - \lambda_{12}^{(0)\,2}\right)^{1/2}}{\lambda_{22}^{(0)}}, \tag{31}$$

$$A_{11} = 1,\ N_{11} = 1,\ L_{1i1} = \lambda_{i1}^{(0)} + \tau_1\lambda_{i2}^{(0)} \text{ for } i = 1, 2.$$

Also the fundamental solution (20) takes the form

$$\Phi_{11}(\mathbf{x}, \mathbf{x_0}) = -\frac{g^{-1/2}(\mathbf{x})g^{-1/2}(\mathbf{x_0})}{2\pi\mathrm{i}\lambda_{22}^{(0)}(\tau_1 - \overline{\tau}_1)}\,\Re\mathfrak{e}\left[\log(z_1 - c_1)\right], \tag{32}$$

where $z_1 = x_1 + \tau_1 x_2$, $c_1 = \xi_1 + \tau_1\xi_2$ and from (15) $g^{1/2}(\mathbf{x})$ satisfies

$$\lambda_{ij}^{(0)}\frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0\ , \tag{33}$$

which has the general solution

$$g(\mathbf{x}) = [\Re\mathfrak{e} F(z_1)]^2\ , \tag{34}$$

where $F(z_1)$ is an arbitrary analytic function of the complex variable z_1. This arbitrary analytic function may be chosen to best approximate given numerical data for the conductivity coefficients.

Setting $\phi_1(\mathbf{x}) = T(\mathbf{x})$ the boundary-integral equation (5) for plane anisotropic thermostatics becomes

$$\lambda T(\mathbf{x_0}) = \int_{\partial\Omega}\left[P_1(\mathbf{x})\Phi_{11}(\mathbf{x}, \mathbf{x_0}) - \Gamma_{11}(\mathbf{x}, \mathbf{x_0})T(\mathbf{x})\right]\mathrm{d}S(\mathbf{x})\ , \tag{35}$$

where Φ_{11} is given by (32) and from (3) and (7) the heat flux P_1 and the function Γ_{11} are given by

$$P_1 = \lambda_{ij}\frac{\partial T}{\partial x_j}n_i, \tag{36}$$

$$\Gamma_{11} = \lambda_{ij}\frac{\partial \Phi_{11}}{\partial x_j}n_i\ . \tag{37}$$

Once the constants $\lambda_{ij}^{(0)}$ are known and the arbitrary analytic function $F(z_1)$ is specified, the $g(\mathbf{x})$ may be obtained from (34) and then (32) and (37) readily yield numerical values of the fundamental solution Φ_{11} and Γ_{11} for any given values of z_1 and c_1.

For anisotropic thermostatics the Green's function given by (27), (20) and (28) takes the form

$$\Phi_{11}(\mathbf{x}, \mathbf{x_0}) = -\frac{g^{-1/2}(\mathbf{x})g^{-1/2}(\mathbf{x_0})}{2\pi \mathrm{i}\lambda_{22}^{(0)}(\tau_1 - \overline{\tau}_1)}\, \Re\mathfrak{e}\left[\log(z_1 - c_1) - \log(z_1 - \overline{c}_1)\right]. \tag{38}$$

This Green's function may be advantageously employed in the integral equation (38) in cases when a substantial section of the boundary $\partial\Omega$ lies on $x_2 = 0$ and along this section of the boundary the temperature T is zero or a non-zero constant. Examples of problems of this type include determining the two dimensional temperature and flux fields around subterranean cavities with the earth's surface being taken to be the plane $x_2 = 0$.

6.2. Antiplane deformations of anisotropic elastic materials

In (1) put $N = 1$, $\phi_1 = u$ and $a_{1i1j} = \lambda_{ij}$ for $i, j = 1, 2$ where, referred to a Cartesian frame $Ox_1x_2x_3$, $u(\mathbf{x})$ denotes the antiplane displacement and the λ_{ij} are the elastic coefficients which satisfy the symmetry property $\lambda_{ij} = \lambda_{ji}$. In this case (1) becomes the equation governing antiplane deformations of inhomogeneous elastic materials. The equation is formally identical to (29) so that the previous analysis for plane anisotropic thermostatics may be employed for antiplane elasticity with the temperature T replaced by the antiplane displacement u, the λ_{ij} interpreted as the elastic coefficients and the P_1 as the antiplane stress. Hence it is possible to use the integral equation (35) together with the fundamental solution (32) to solve a wide class of antiplane elastic boundary-value problems for inhomogeneous materials for which the elastic coefficients $\lambda_{ij}(\mathbf{x})$ vary with position according to the Equation (30). Also the Green's function (38) may be used in the integral equation to solve the class of antiplane elastic problems for which the elastic material adheres to a rigid plane surface which coincides with the $x_2 = 0$ plane.

6.3. Generalised plane deformations of anisotropic elastic materials

In (1) put $N = 3$, $\phi_i = u_i$ and $a_{ijkl} = c_{ijkl}$ for $i, j, k, l = 1, 2, 3$ where, referred to a coordinate frame $Ox_1x_2x_3$, $u_i(\mathbf{x})$ denotes the displacement and the c_{ijkl} are the elastic coefficients which satisfy the symmetry property $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} = c_{ilkj}$. Further, they vary with position according to the equation

$$c_{ijkl}(\mathbf{x}) = c_{ijkl}^{(0)} g(\mathbf{x}), \tag{39}$$

where the $c_{ijkl}^{(0)}$ are constants. Thus with $N = 3$ and the dependent variable and coefficients so interpreted the analysis of Sections 2 through 5 may be applied to obtain a boundary-integral equation which is suitable for the numerical solution of generalised plane boundary-value problems for the class of inhomogeneous elastic materials that satisfy the above symmetry conditions and the Equation (39).

For generalised plane elastostatic problems the Green's function (38) could be usefully employed for problems of this type involving deformations of materials which adhere to a rigid plane boundary. Examples include deformations of an elastic layer adhering to a rigid foundation where the interface is taken to be the plane $x_2 = 0$.

7. Final remarks

A fundamental solution and a Green's function have been obtained for a system of N second-order elliptic partial differential equations in N dependent and two independent variables. The variable coefficients in the equations are each a multiple of the solution of a system of N second-order elliptic partial differential equations in one dependent and two independent variables. The fundamental solution and the Green's function are in forms which readily yield numerical values and hence are suitable for use in the relevant boundary-integral equation for the purpose of obtaining the numerical solution to a wide class of particular boundary-value problems.

It should be noted that a number of authors have considered the use of boundary-integral equations for the solution of problems governed by particular partial differential equations which are special cases of the general system (1). For Darcy's flow Cheng [4] directly uses the relevant special case of the fundamental solution given by (20) and (21) to facilitate the numerical solution of a number of boundary-value problems. Other examples involving special cases of (1) are the papers of Clements and Budhi [5], Azis and Clements [6], Ang, Kusuma and Clements [7], Kessab and Divo [8], Shaw [9] and Shaw and Gipson [10]. In these papers various techniques are employed to obtain a suitable boundary-integral equation together with the associated fundamental solution for problems governed by a either a single or a system of two second-order elliptic equation with variable coefficients. In some of these cases the equation under consideration falls within the general class given by (1) but the coefficients are not constrained by the condition (8). In such cases the fundamental solution, while being relevant for a more general class of coefficients, is much more complex and difficult to use for numerical computations than the fundamental solution (20).

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